Assignment 8

This homework is due Friday March 27.

This assignment is *longer than usual*. It is worth $\frac{3}{2}$ of a usual homework in terms of course grade.

There are total 65 points in this assignment. 58 points is considered 100%. If you go over 58 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 5.2–5.5 of Textbook.

(1) [10pt] Find all values of the following. (Reminder: $\log z$ is a multivalued function, $\operatorname{Log} z$ is its principal branch.)

(a) $\operatorname{Log}(ie^2)$,
(b) $\operatorname{Log}(\sqrt{3} - i)$,
(c) $\operatorname{Log}((1+i)^4)$,
(d) $\log(-3)$,
(e) $\log(-\sqrt{2} + i\sqrt{2})$,
(f) $\cos(1+i)$,
(g) $\sin 2i$,
(h) $\tan\left(\frac{\pi+i}{2}\right)$,
(i) $\cosh\left(\frac{4-i\pi}{4}\right)$,
(j) $\sinh(1+i\pi)$.

(2) [5pt]

(a) For what values of z_1, z_2 is it true that $\operatorname{Log}\left(\frac{z_1}{z_2}\right) = \operatorname{Log}(z_1) - \operatorname{Log}(z_2)$? Why?
(b) Give an example of specific values of z_1, z_2 such that $\operatorname{Log}\left(\frac{z_1}{z_2}\right) \neq \operatorname{Log}(z_1) - \operatorname{Log}(z_2)$.

(3) [5pt] Use analyticity of $\operatorname{Log} z$ in the right half-plane $\operatorname{Re}(z) > 0$ to show that the functions $u(x,y) = \ln(x^2+y^2)$ and $v(x,y) = \tan^{-1}(y/x)$ are harmonic in the same domain.

(4) [5pt] Solve the following equations (i.e. find all possible values of z).

(a) $\operatorname{Log}(z) = 1 - i\frac{\pi}{4}$.
(b) $\operatorname{Log}(z-1) = i\frac{\pi}{2}$.
(c) $e^{iz} = -1$.
(d) $\exp(z+1) = i$.

(5) [4pt] Find the principal value of
(a) 4^i.
(b) $(-1)^{\frac{1}{\pi}}$.
(c) $(1+i\sqrt{3})^{\frac{i}{2}}$.

(6) [4pt] Find all values of
(a) i^i.
(b) $(-1)^{\sqrt{2}}$.
(c) $(-1)^{\frac{3}{4}}$.
In which of the above cases are there infinitely many values?

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(7) [5pt] For $z = re^{i\theta} \neq 0$, show that for $r > 0$ and $-\pi < \theta \leq \pi$, the principal branch of the function
  (a) z^i is given by $z^i = e^{-\theta}(\cos(\ln r) + i\sin(\ln r))$.
  (b) z^α (with real α) is given by $z^\alpha = r^\alpha(\cos\alpha\theta + i\sin\alpha\theta)$.
  (*Hint:* Use the definition of the power function.)

(8) [7pt] Use expression of trigonometric and hyperbolic functions through the exponential function to establish the following:
  (a) $\sin\left(\frac{\pi}{2} - z\right) = \cos z$.
  (b) $\tanh(z + i\pi) = \tanh z$.
  (c) $\cos 2z = \cos^2 z - \sin^2 z$.
  (d) Find a similar formula for $\cosh 2z$.

(9) [5pt] Show that $\sin\overline{z} = \overline{\sin z}$ and that $\sin\overline{z}$ is nowhere analytic.

(10) [8pt] Find *all* values of the following. (Express them as $x + iy$.)
  (a) $\arcsin\frac{\sqrt{3}}{2}$.
  (b) $\arcsin 3$.
  (c) $\arccos 3i$.
  (d) $\arctan i$.
  (e) $\arctan 2i$.

(11) [7pt] Show that
  (a) $\arccos z = -i\log\left(z + i(1-z^2)^{\frac{1}{2}}\right)$.
  (b) $\operatorname{arcsinh} z = \log\left(z + (1+z^2)^{\frac{1}{2}}\right)$.
  (c) $\frac{d}{dz}\arccos z = \frac{-1}{(1-z^2)^{\frac{1}{2}}}$.